ACADEMIC PRESS
Available online at www.sciencedirect.com
SCIENCE DIRECT
JOURNAL OF Algebra
Journal of Algebra 256 (2002) 28–50
www.academicpress.com

Classification of division $\mathbb{Z}^n$-graded alternative algebras

Yoji Yoshii [*,1]

Department of Mathematical Sciences, University of Alberta, Edmonton, AL, Canada T6G 2G1

Received 24 January 2001

Communicated by Alexander Premet

Dedicated to the memory of my wife Fumiko

Abstract

The octonion torus (or Cayley torus) appears as a coordinate algebra of extended affine Lie algebras of type A_2 and F_4. A generalized version of the octonion torus, called *division* $\mathbb{Z}^n$*-graded alternative algebras*, is classified in this paper. Using the result, we can complete the classification of division $(A_2, \mathbb{Z}^n)$-graded Lie algebras, up to central extensions, which are a generalization of the cores of extended affine Lie algebras of type A_2.

Introduction

In this paper we classify division $\mathbb{Z}^n$-graded alternative algebras. It turns out that they are strongly prime, and so one can apply Slater's Theorem classifying prime alternative algebras [1]. Namely, a strongly prime alternative algebra is either associative or an *octonion ring*; i.e., its central closure is an octonion

* Present address: Department of Mathematics, University of Wisconsin–Madison, 480 Lincoln Dr., Van Vleck Hall, Madison, WI 53706, USA.

E-mail address: yoshii@math.wisc.edu.

1 Research supported by a PIMS postdoctoral fellowship (2001) and an NSERC postdoctoral fellowship (2002).

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00022-4

algebra. In [2] division $\mathbb{Z}^n$-graded associative algebras were classified. Therefore, our purpose here is to classify division $\mathbb{Z}^n$-graded octonion rings.

Let us present four such octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and $\mathbb{O}_4$. Let F be a field of characteristic $\neq 2$ and K any field extension of F. We define

$$\mathbb{O}_i = \left(K\left[t_1^{\pm 1}, \ldots, t_n^{\pm 1}\right], \mu_1, \mu_2, \mu_3\right);$$

i.e., the octonion algebra over $K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ obtained by the Cayley–Dickson process with the structure constants μ_1, μ_2, and μ_3, for $i = 1, \ldots, 4$:

[$i = 1$]: $0 \neq \mu_1, \mu_2, \mu_3 \in K$ such that (K, μ_1, μ_2, μ_3) is an octonion division algebra;
[$i = 2$]: $0 \neq \mu_1, \mu_2 \in K$, and $\mu_3 = t_1$ such that (K, μ_1, μ_2) is a quaternion division algebra;
[$i = 3$]: $0 \neq \mu_1 \in K$, $\mu_2 = t_1$, and $\mu_3 = t_2$ such that (K, μ_1) is a field;
[$i = 4$]: $\mu_1 = t_1$, $\mu_2 = t_2$, and $\mu_3 = t_3$.

Note that $\mathbb{O}_4$ is the Cayley torus over K, which is a coordinate algebra of extended affine Lie algebras of A_2 and F_4 (see [3,4]). Our main result is the following theorem.

Theorem. *A division $\mathbb{Z}^n$-graded octonion ring is isomorphic to exactly one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$, and $\mathbb{O}_4$.*

As a corollary, we obtain the classification of alternative tori over any field, which generalizes a result in [3]. Namely, an alternative torus is isomorphic to a quantum torus $F_{\boldsymbol{q}}$ or the Cayley torus $\mathbb{O}_t = (F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], t_1, t_2, t_3)$. Moreover, with a result in [2], a division $\mathbb{Z}^n$-graded alternative algebra is isomorphic to a division $\mathbb{Z}^n$-graded associative algebra $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ (a natural generalization of a quantum torus $F_{\boldsymbol{q}}$) or the octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$, and $\mathbb{O}_4$, which completes the classification. Finally, one can also classify division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebras, a generalized concept of the core of an extended affine Lie algebra of type A_2, introduced in [2].

The organization of the paper is as follows. In Section 1 we review alternative algebras. In Section 2 we summarize some basic properties of graded algebras. In Section 3 we review the Cayley–Dickson process over a ring. In Section 4 quantum tori of degree 2 are classified. In Section 5 we show that the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$, and $\mathbb{O}_4$ described above are in fact non-isomorphic division $\mathbb{Z}^n$-graded alternative algebras, and prove that these four exhaust the division $\mathbb{Z}^n$-graded alternative but not associative algebras. Finally, the classification of division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebras is stated in Section 6.

1. Review of alternative algebras

Throughout let F be a field and assume for this section that all algebras are unital and *alternative*; i.e., they satisfy the identities $(x,x,y) = 0 = (x,y,y)$, where the *associator* $(x,y,z) = (xy)z - x(yz)$. It is easy to check the following identity $(x,y,x) = 0$, called the *flexible law*. Thus we denote $xyx := (xy)x = x(yx)$.

We recall some basic notions.

Definition 1.1. (1) Let A be an algebra and $0 \neq x \in A$. Then x is called a *zero-divisor* if $xy = 0$ or $yx = 0$ for some $0 \neq y \in A$.

(2) A is called

(i) a *domain* if there is no zero-divisor in A,
(ii) *non-degenerate* if $xAx = 0$ implies $x = 0$ for all $x \in A$ where $xAx = \{xax \mid a \in A\}$,
(iii) *prime* if $IK = (0)$ implies $I = (0)$ or $K = (0)$ for all ideals I, K of A where $IK = \{\sum_{x,y} xy \mid x \in I,\ y \in K\}$,
(iv) *strongly prime* if A is non-degenerate and prime.

The non-degeneracy and primeness generalize the notion of a domain.

Lemma 1.2. *A domain is strongly prime.*

Proof. Straightforward. □

For an algebra A, the *centre* $Z(A)$ of A is defined as

$$Z(A) = \bigl\{z \in A \bigm| [z,x] = (z,x,y) = 0 \text{ for all } x,y \in A\bigr\},$$

where the *commutator* $[x,y] = xy - yx$. The centre is always a commutative associative subalgebra and any algebra can be considered as an algebra over the centre.

Definition 1.3. For an algebra A with the property that the centre $Z = Z(A)$ is an integral domain (e.g., a prime algebra), we define $\overline{A}$ over $\overline{Z}$ as $\overline{A} = \overline{Z} \otimes_Z A$ where $\overline{Z}$ is the field of fractions of Z, and call $\overline{A}$ the *central closure of* A.

2. Division G-graded alternative algebras

Throughout this section let G be a group and all algebras be unital and alternative over a field F. An algebra $A = \bigoplus_{g \in G} A_g$ is called *G-graded* if $A_g A_h \subset A_{gh}$ for all $g,h \in G$. We will refer to A_g as a *homogeneous space* and an element of A_g as a *homogeneous element of degree* g. A graded algebra is called *strongly graded* if $A_h A_g = A_{hg}$ for all $h, g \in G$ and *division graded* if any non-zero element of every homogeneous space is invertible.

For a G-graded algebra $A = \bigoplus_{g\in G} A_g$, we note that $A = \bigoplus_{g\in\langle \operatorname{supp} A\rangle} A_g$, where $\operatorname{supp} A := \{g \in G \mid A_g \neq (0)\}$ and $\langle \operatorname{supp} A\rangle$ is the subgroup of G generated by $\operatorname{supp} A$. From now on,

we always assume that $\langle \operatorname{supp} A\rangle = G$.

Thus, if A is division graded, then this assumption is equivalent to saying that $\operatorname{supp} A = G$, or every homogeneous space is non-zero.

The following lemma is straightforward to check.

Lemma 2.1. *If G is a totally ordered group, then a division G-graded algebra is a domain.*

In general, a subalgebra H of a graded algebra $A = \bigoplus_{g\in G} A_g$ is called *homogeneous* if $H = \bigoplus_{g\in G}(H \cap A_g)$. If G is abelian, the centre Z is homogeneous. Moreover, if A is division graded, then $\Gamma := \{g \in G \mid Z \cap A_g \neq (0)\}$ is a subgroup of G and Z is a division Γ-graded commutative associative algebra. We call this Γ the *central grading group of* A.

Lemma 2.2. *Let $Z = \bigoplus_{g\in G} Z_g$ be a division G-graded commutative associative algebra over F. Let D be a division algebra over the field $K := Z_e$ (e is the identity element of G). Then $Z \otimes_K D = \bigoplus_{g\in G}(Z_g \otimes_K D)$ is a division G-graded algebra over K.*

Proof. Straightforward. □

A division G-graded algebra is a graded Z-module of type G; and so one can show the following lemma (see Theorem 3 and Corollary 2 in [5, pp. 29–30]).

Lemma 2.3. *Let $A = \bigoplus_{g\in G} A_g$ be a division G-graded algebra over F for an* abelian group G. *Let Z be the centre of A and Γ the central grading group of A; and so $Z = \bigoplus_{h\in\Gamma} Z_h$ for $Z_h = Z \cap A_h$. Let $K := Z_e$ which is a field. Then*:

(i) $A_{hg} = zA_g = Z_h A_g$ *for all* $h \in \Gamma$, $g \in G$ *and any* $0 \neq z \in Z_h$.
(ii) *For* $\bar{g} \in G/\Gamma$, *we have* $ZA_g = \bigoplus_{g'\in\bar{g}} A_{g'}$ *and, in particular, if* $\bar{g} = \overline{g'}$, *then* $ZA_g = ZA_{g'}$.
(iii) *Let* $A_{\bar{g}} := ZA_g$. *Then* $A = \bigoplus_{\bar{g}\in G/\Gamma} A_{\bar{g}}$ *and A is a G/Γ-graded algebra.*
(iv) *For* $g \in G$, $A_{\bar{g}}$ *is a free Z-module and any K-basis of the K-vector space* A_g *becomes a Z-basis of* $A_{\bar{g}}$; *and so* $\operatorname{rank}_Z A_{\bar{g}} = \dim_K A_{g'}$ *for all* $\bar{g} \in G/\Gamma$ *and* $g' \in \bar{g}$.

Let G be a totally ordered abelian group. Then a division G-graded algebra is a domain by Lemma 2.1; and so the central closure $\overline{A}$ makes sense (see Definition 1.3). Also, we identify A with a subalgebra of $\overline{A}$ via $a \mapsto 1 \otimes a$. Thus we have a similar lemma.

Lemma 2.4. *In the notation of Lemma* 2.3, *let $\overline{A} = \overline{Z} \otimes_Z A$ be the central closure and $\overline{A}_{\bar{g}} := \overline{Z} \otimes_Z A_{\bar{g}}$ for $\bar{g} \in G/\Gamma$. Then*:

(i) $\overline{A} = \bigoplus_{\bar{g} \in G/\Gamma} \overline{A}_{\bar{g}}$ *and $\overline{A}$ is a G/Γ-graded algebra.*
(ii) *Any K-basis of A_g becomes a $\overline{Z}$-basis of $\overline{A}_{\bar{g}}$; and hence* $\dim_{\overline{Z}} \overline{A}_{\bar{g}} = \dim_K A_{g'}$ *for all $\bar{g} \in G/\Gamma$ and $g' \in \bar{g}$. Also, there exists a $\overline{Z}$-basis of $\overline{A}$ which is a Z-basis of A.*

We now define a special type of G-graded algebras for any group G.

Definition 2.5. A division G-graded algebra $T = \bigoplus_{g \in G} T_g$ is called an *alternative G-torus over F* if $\dim_F T_g = 1$ for all $g \in G$. Moreover, if T is associative, it is called an *associative G-torus*.

Strongly-graded does not imply division-graded (see [6, Exercise 3, p. 18]). However, we have the following statement.

Lemma 2.6. *A G-graded algebra $T = \bigoplus_{g \in G} T_g$ is an alternative G-torus if and only if T is strongly graded and* $\dim_F T_g = 1$ *for all $g \in G$.*

Proof. We only need to show the if part. Let $0 \neq x \in T_g$. Then we have $F1 = T_e = T_g T_{g^{-1}} = F x T_{g^{-1}}$. So there exists a non-zero element $u \in T_{g^{-1}}$ such that $xu = 1$. Since $ux \in T_e = F1$, we have $ux = c1$ for some $c \in F$. Then, by the flexible law, we get $x = (xu)x = x(ux) = cx$. Hence $c = 1$, i.e., $xu = ux = 1$. Therefore x is invertible. $\square$

By Lemma 2.4, the central closure of an alternative G-torus is an alternative G/Γ-torus for Γ the central grading group if G is a totally ordered abelian group.

Definition 2.7. An alternative (respectively associative) $\mathbb{Z}^n$-torus is called an *alternative* (respectively *associative*) *n-torus over F* or simply an *alternative* (respectively *associative*) *torus* when this abbreviation does not create confusion.

We recall quantum tori (see, for example, [7] or [8]). An $n \times n$ matrix $\boldsymbol{q} = (q_{ij})$ over F such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix*. The *quantum torus* $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ *determined by* a quantum matrix $\boldsymbol{q}$ is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$

satisfying relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leqslant i, j \leqslant n$. Note that $F_{\boldsymbol{q}}$ is commutative if and only if $\boldsymbol{q} = \mathbf{1}$ where $\mathbf{1}$ is the matrix whose entries are all 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra of Laurent polynomials $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. One can give a $\mathbb{Z}^n$-grading of the quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ in an obvious way: For any basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ of $\mathbb{Z}^n$, we define the degree of

$$t_{\boldsymbol{\alpha}} := t_1^{\alpha_1} \cdots t_n^{\alpha_n} \quad \text{for } \boldsymbol{\alpha} = \alpha_1 \boldsymbol{\sigma}_1 + \cdots + \alpha_n \boldsymbol{\sigma}_n \in \mathbb{Z}^n \text{ as } \boldsymbol{\alpha}.$$

Then $F_{\boldsymbol{q}} = \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} F t_{\boldsymbol{\alpha}}$ appears a $\mathbb{Z}^n$-graded algebra and an associative torus. We call this grading the *toral $\mathbb{Z}^n$-grading of $F_{\boldsymbol{q}}$ determined by $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$*. Sometimes it is referred to as a *$\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading*. We always assume some toral $\mathbb{Z}^n$-grading of $F_{\boldsymbol{q}}$. Conversely, any associative torus is graded isomorphic to some $F_{\boldsymbol{q}}$ with some toral grading (see Lemma 1.8 in [3]).

By Artin's Theorem, the subalgebra of an algebra generated by two elements is associative [1, p. 36]. Hence an alternative 1-torus is isomorphic to $F[t, t^{-1}]$.

3. Cayley–Dickson process over a ring

We review the Cayley–Dickson process over a ring [9, p. 103]. In this section, our algebras are arbitrary non-associative unital algebras, not necessarily alternative, over a ring of scalars Φ (a commutative associative unital ring). Moreover, for an algebra B over Φ, we assume that B is *faithful*; i.e., for all $\alpha \in \Phi$, $\alpha B = 0 \Longrightarrow \alpha = 0$. Since B has a unit, B is faithful if and only if, for all $\alpha \in \Phi$, $\alpha 1 = 0 \Longrightarrow \alpha = 0$. Let $*$ be a *scalar involution over Φ*, i.e., an anti-automorphism of period 2 with $bb^* \in \Phi 1$ (hence by linearization, $b + b^* \in \Phi 1$) for all $b \in B$; and let $\mu \in \Phi$ be a *cancellable scalar*, i.e., $\mu b = 0$ for some $b \in B \Longrightarrow b = 0$. Then one can construct a new algebra $B \oplus B$ with its product $(a, b)(c, d) = (ac + \mu d b^*, a^* d + cb)$ for $a, b, c, d \in B$. Letting $v = (0, 1)$ we can write this algebra as $B \oplus vB$:

$$(a + vb)(c + vd) = (ac + \mu d b^*) + v(a^* d + cb). \tag{3.1}$$

We say that this new algebra $B \oplus vB$ is obtained from $B = (B, *)$ by the *Cayley–Dickson process with structure constant μ* and denote it by (B, μ). We define a new map $*$ on $C := (B, \mu)$ by

$$(a + vb)^* = a^* - vb. \tag{3.2}$$

This new $*$ is also a scalar involution of C. Hence, for a cancellable scalar ν, one can obtain another new algebra (C, ν) from $C = (C, *)$ by the Cayley–Dickson process with structure constant ν. We write this (C, ν) as (B, μ, ν) instead of $((B, \mu), \nu))$.

Remark 3.3. (1) Since μ is cancellable, we have $vb = 0 \Longrightarrow b = 0$.

(2) If μ is invertible, then μ is cancellable and $v = (0, 1)$ is invertible in the sense that there exists an element $u \in C$ such that $uv = vu = 1$.

We will use the following lemma from [9, Theorem 6.8, p. 105].

Lemma 3.5. *For the algebra (B, μ), we have*

(i) (B, μ) *is commutative* $\Leftrightarrow$ B *is commutative with trivial involution.*
(ii) (B, μ) *is associative* $\Leftrightarrow$ B *is commutative and associative.*
(iii) (B, μ) *is alternative* $\Leftrightarrow$ B *is associative.*

When Φ is a field of characteristic $\neq 2$ and we start the Cayley–Dickson process from $\Phi = (\Phi, \mathrm{id})$ where id is the trivial scalar involution, i.e., the identity map, we call Φ, (Φ, μ_1), (Φ, μ_1, μ_2), and $(\Phi, \mu_1, \mu_2, \mu_3)$, where μ_1, μ_2, and μ_3 are any non-zero elements in Φ, *composition algebras* over Φ. When Φ is a field of characteristic $= 2$, we define a separable quadratic algebra Φ_μ for $0 \neq \mu \in \Phi$ by $\Phi_\mu := \Phi[X]/(X^2 - X - \mu)$. Then, Φ_μ has a scalar involution over Φ. For convenience, we denote (Φ, μ) for Φ_μ and call $\Phi_{\mu_1} = (\Phi, \mu_1)$, (Φ, μ_1, μ_2), and $(\Phi, \mu_1, \mu_2, \mu_3)$, where $\mu_1, \mu_2, \mu_3 \in \Phi$ and are any nonzero elements, *composition algebras* (see [10]). Also, (Φ, μ_1, μ_2) is called a *quaternion algebra* and $(\Phi, \mu_1, \mu_2, \mu_3)$ an *octonion algebra*.

For an algebra $B = (B, *)$ with scalar involution $*$ and $x \in B$, we define the *norm* $n(a)$ of a as the unique scalar $aa^* = n(a)1$. Also, for $a, b \in B$, we define $n(a, b) := n(a + b) - n(a) - n(b)$; and so $n(a, b)1 = ab^* + ba^*$. Note that $n(\cdot, \cdot)$ is a symmetric bilinear form on B. In the same way, we have the norm n and the symmetric bilinear form $n(\cdot, \cdot)$ on the algebra (B, μ) obtained by the Cayley–Dickson process. For $(B, \mu) = B \oplus vB$ and $a, b \in B$, we have $n(a, vb) = a(vb)^* + (vb)a^* = -a(vb) + (vb)a^* = -v(a^*b) + v(a^*b) = 0$; and so

$$n(B, vB) = 0 \quad \text{or equivalently} \quad vB \subset B^{\perp}, \tag{3.6}$$

where $B^{\perp}$ is the orthogonal submodule relative to $n(\cdot\,, \cdot)$.

We recall that an algebra B, in general, has degree 2 if there exist a linear form tr called *trace* and a quadratic form n called *norm* such that for all $a \in B$

$$a^2 - \mathrm{tr}(a)a + n(a)1 = 0, \qquad \mathrm{tr}(1) = 2, \qquad n(1) = 1.$$

In particular, if Φ is a field, then the trace tr and norm n are unique [9, p. 90]. If an algebra B over Φ has a scalar involution $*$, then one can easily check that B has degree 2, that $a + a^* = \mathrm{tr}(a)1$ for $a \in B$ and that the two norms from $*$ and the degree 2-algebra coincide. In particular, if Φ is a field, then a scalar involution is unique. We state this as a lemma.

Lemma 3.7. *Let B be an algebra with scalar involution $*$ over a field. Then, B has degree 2 and $*$ is the only scalar involution. We have $a^* = \mathrm{tr}(a)1 - a$ for all $a \in B$, where tr is the trace of the degree-2 algebra B.*

We will use the following known lemma later (see [1, p. 32]).

Lemma 3.8. *Let $C = (C, *)$ be a division composition algebra over a field Φ. Let B be a subalgebra of C such that $\dim_\Phi B = \frac{1}{2} \dim_\Phi C$. Then, B is a division composition algebra with scalar involution $* = *|_B$ over Φ and $C = (B, \mu)$ for some $0 \neq \mu \in \Phi$, except when $\mathrm{ch}\, \Phi = 2$ and $\dim_\Phi C = 2$.*

4. Quantum tori of degree 2

We will classify quantum tori (= associative tori) of degree 2 in Proposition 4.3. First we show some properties for a certain G-graded alternative algebra which has a scalar involution. Essentially they are shown in [3, Lemma 1.24] for the case of alternative tori, but for the convenience of the reader, we show this result.

Lemma 4.1. *Let $A = \sum_{g \in G} A_g$ be a G-graded alternative algebra over a field L which has a scalar involution $*$. Assume that the non-zero homogeneous elements are not nilpotents. Then we have the following*:

(i) *For any $x \in A_g$, $g \neq e$, we have $\mathrm{tr}(x) = 0$, $x^2 = -n(x)1$, $x^* = -x$ and $*$ is graded; i.e., $A_g^* = A_g$ for all $g \in G$.*

(ii) *The exponent of G is 2 or G is trivial; i.e., $G = \{e\}$.*
In particular, if $G = \mathbb{Z}^n/\Gamma$ for some subgroup Γ of $\mathbb{Z}^n$, there exists a basis $\{\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n\}$ of $\mathbb{Z}^n$ and $m \geqslant 0$ such that

$$\Gamma = 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\varepsilon}_m + \mathbb{Z}\boldsymbol{\varepsilon}_{m+1} + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n.$$

(iii) *All homogeneous spaces are orthogonal to each other relative to the norm.*

Proof. For (i) and (ii), let $0 \neq x \in A_g$. Then $x^2 \neq 0$ by our assumption. Since A has degree 2, we have $x^2 + n(x)1 = \mathrm{tr}(x)x \in A_g$. If $\mathrm{tr}(x) \neq 0$, then $g^2 = g$ since $x^2 \neq 0$. Hence $g = e$, which contradicts our assumption. So we get $\mathrm{tr}(x) = 0$. Then $x^2 = -n(x)1$. Moreover, $0 \neq x^2 \in A_{g^2} \cap L1 \subset A_{g^2} \cap A_e$ and hence $g^2 = e$. The second statement of (ii) follows from Fundamental Theorem of finitely generated abelian groups. By Lemma 3.7, we get $x^* = \mathrm{tr}(x)1 - x = -x$. For $y \in A_e$, we have $y^* = \mathrm{tr}(y)1 - y \in A_e$; and so $*$ is graded.

For (iii), let $y \in A_h$ where $g \neq h \in G$. Then we have $xy^* \in A_{gh}$ since $*$ is graded. If $gh = e$, then $g = h$ since the exponent of G is 2. Thus $gh \neq e$ and by (i); one gets $\mathrm{tr}(xy^*)1 = 0$. Hence, $n(x, y)1 = xy^* + yx^* = \mathrm{tr}(xy^*) = 0$. $\square$

Example 4.2. Let $\operatorname{ch} F \neq 2$ and let $F_{\boldsymbol{h}} = F_{\boldsymbol{h}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the quantum torus determined by $\boldsymbol{h} = (h_{ij})$ where the $h_{12} = h_{21} = -1$ and the other entries are all 1, and Z the centre of $F_{\boldsymbol{h}}$. One finds that $Z = F[t_1^{\pm 2}, t_2^{\pm 2}, t_3^{\pm 1}, \ldots, t_n^{\pm 1}]$, the algebra of Laurent polynomials in the variables $t_1^2, t_2^2, t_3, \ldots, t_n$. So for a $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\boldsymbol{h}}$, the central grading group of $F_{\boldsymbol{h}}$ is $2\mathbb{Z}\boldsymbol{\sigma}_1 + 2\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$ and the central closure $\overline{F}_{\boldsymbol{h}}$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-torus over $\overline{Z}$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Thus the dimension of $\overline{F}_{\boldsymbol{h}}$ over $\overline{Z}$ is 4. Since $\overline{F}_{\boldsymbol{h}}$ has no zero divisors by Lemma 2.1 and is finite dimensional, it is a division algebra. Hence $\overline{F}_{\boldsymbol{h}}$ is a quaternion division algebra and has degree 2 over $\overline{Z}$. Clearly, $\overline{F}_{\boldsymbol{h}} = (\overline{Z}, t_1^2, t_2^2)$ generated by t_1, t_2, and $*: t_1 \mapsto -t_1$, $t_2 \mapsto -t_2$ defines a scalar involution of $\overline{F}_{\boldsymbol{h}}$. Since $(F_{\boldsymbol{h}})^* = F_{\boldsymbol{h}}$ and $F_{\boldsymbol{h}} \cap \overline{Z} = Z$, the restricted involution $* := *|_{F_{\boldsymbol{h}}}$ is scalar over Z. Thus $F_{\boldsymbol{h}}$ has degree 2 over Z and $(F_{\boldsymbol{h}}, *) = (Z, t_1^2, t_2^2)$. We call this $F_{\boldsymbol{h}}$ a *quaternion torus*.

Proposition 4.3. *Let T be a non-commutative associative torus over F and Γ the central grading group of T. Then the following are equivalent*:

(i) $\operatorname{ch} F \neq 2$ *and T is graded isomorphic to the quaternion torus for some toral grading*;
(ii) *T has degree* 2 *over Z*;
(iii) *$\overline{T}$ has degree* 2 *over $\overline{Z}$*;
(iv) *$\Gamma = 2\mathbb{Z}\boldsymbol{\sigma}_1 + 2\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$ for some toral $\{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n\}$-grading*;
(v) *T has a scalar involution over Z*;
(vi) *$\overline{T}$ has a scalar involution over $\overline{Z}$.*

Proof. We already showed (i) $\Rightarrow$ (ii) in Example 4.2. (ii) $\Rightarrow$ (iii) is clear. For (iii) $\Rightarrow$ (iv), we have $\dim_{\overline{Z}} \overline{T} = 2^m$ by Lemma 4.1(ii). But $\overline{T}$ is a finite-dimensional central associative division algebra over $\overline{Z}$; and so $m = 2$. (iv) $\Rightarrow$ (i) can be shown in the same way as Proposition 6.12 in [11] (non-commutativity of T forces $\operatorname{ch} F \neq 2$). Thus we get (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

We already showed (i) $\Rightarrow$ (v) in Example 4.2. (v) $\Rightarrow$ (vi) is clear. Since (vi) $\Rightarrow$ (iii), we obtain (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). □

5. Division $\mathbb{Z}^n$-graded alternative algebras

In this section all algebras are unital over a field F. A prime alternative algebra A is called an *octonion ring* if the central closure $\overline{A}$ is an octonion algebra over the field $\overline{Z}$ [1, p. 193]. We first state Slater's Theorem [1, Theorem 9, p. 194].

Theorem 5.1. *A strongly prime alternative algebra over F is either an associative algebra or an octonion ring.*

Thus:

Theorem 5.2. *A division G-graded alternative algebra A over F for a totally ordered group G is either a division G-graded associative domain or octonion ring which embeds into an octonion division algebra over $\overline{Z}$.*

Proof. By Lemma 2.1, A is a domain. Hence by Theorem 5.1, it is either a division G-graded associative domain or octonion ring. Finally, $\overline{A}$ is a domain and hence $\overline{A}$ is a division algebra (see [1, Lemma 9, p. 43]). $\square$

Now, we specify $G = \mathbb{Z}^n$ and classify division $\mathbb{Z}^n$-graded alternative algebras. Since $\mathbb{Z}^n$ is a totally ordered abelian group, we can apply Theorem 5.2. We classify division $\mathbb{Z}^n$-graded alternative algebras over F which are not associative. By Theorem 5.2, any such algebra is an octonion ring which embeds into an octonion division algebra. We first construct four examples of such octonion rings. For this purpose, we prove a lemma and its corollary.

Lemma 5.3. *Let $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$ be a basis of a free abelian group Λ of rank n and*

$$\Gamma_i := \mathbb{Z}\boldsymbol{\varepsilon}_1 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_{i-1} + 2\mathbb{Z}\boldsymbol{\varepsilon}_i + \mathbb{Z}\boldsymbol{\varepsilon}_{i+1} + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n.$$

Let $A = \bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} A_{\boldsymbol{\alpha}}$ be a division Γ_i-graded associative algebra over a field K and R a subalgebra contained in the centre of A. Suppose

(i) *A has a scalar involution $*$ over R which is graded: $A_{\boldsymbol{\alpha}}^* = A_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \Gamma_i$;*
(ii) *there exists $z \in A_{2\boldsymbol{\varepsilon}_i} \cap R$ such that z is invertible in R.*

Then the algebra (A, z) obtained by the Cayley–Dickson process over R has a unique Λ-grading such that every homogeneous space of degree $\boldsymbol{\alpha} \in \Gamma_i$ in (A, z) is equal to $A_{\boldsymbol{\alpha}}$ and $v = (0, 1) \in (A, z)$ has degree $\boldsymbol{\varepsilon}_i$. Moreover, by the Λ-grading, (A, z) becomes a division Λ-graded alternative algebra over K and the scalar involution $*$ of (A, z) is again graded.

Proof. We have

$$(A, z) = A \oplus vA = \left(\bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} A_{\boldsymbol{\alpha}}\right) \oplus v\left(\bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} A_{\boldsymbol{\alpha}}\right).$$

We first claim $v(\bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} A_{\boldsymbol{\alpha}}) = \bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} vA_{\boldsymbol{\alpha}}$. We then have $v(\bigoplus_{\boldsymbol{\alpha} \in \Gamma_i} A_{\boldsymbol{\alpha}}) = \sum_{\boldsymbol{\alpha} \in \Gamma_i} vA_{\boldsymbol{\alpha}}$; and so we only need to show the sum in the right-hand side is direct. By Remark 3.3(1), $\sum_{\boldsymbol{\alpha} \in \Gamma_i} vx_{\boldsymbol{\alpha}} = v(\sum_{\boldsymbol{\alpha} \in \Gamma_i} x_{\boldsymbol{\alpha}}) = 0$ for $x_{\boldsymbol{\alpha}} \in A_{\boldsymbol{\alpha}}$ implies $\sum_{\boldsymbol{\alpha} \in \Gamma_i} x_{\boldsymbol{\alpha}} = 0$. So we get $x_{\boldsymbol{\alpha}} = 0$ for all $\boldsymbol{\alpha} \in \Gamma_i$ and our claim is settled. For $\boldsymbol{\alpha} \in \Lambda$, define

$$A'_{\boldsymbol{\alpha}} := \begin{cases} A_{\boldsymbol{\alpha}}, & \text{if } \boldsymbol{\alpha} \in \Gamma_i, \\ vA_{\boldsymbol{\alpha} - \boldsymbol{\varepsilon}_i}, & \text{otherwise.} \end{cases}$$

Then we get

$$(A,z)=\Big(\bigoplus_{\boldsymbol{\alpha}\in\Gamma_i}A_{\boldsymbol{\alpha}}\Big)\oplus\Big(\bigoplus_{\boldsymbol{\alpha}\in\Gamma_i}vA_{\boldsymbol{\alpha}}\Big)=\bigoplus_{\boldsymbol{\alpha}\in\Lambda}A'_{\boldsymbol{\alpha}},$$

the direct sum of F-vector spaces. By the multiplication of (A,z), we have

$$A'_{\boldsymbol{\alpha}}A'_{\boldsymbol{\beta}}=\begin{cases} A_{\boldsymbol{\alpha}}A_{\boldsymbol{\beta}}=A_{\boldsymbol{\alpha}+\boldsymbol{\beta}}=A'_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \\ \quad \text{if } \boldsymbol{\alpha},\boldsymbol{\beta}\in\Gamma_i, \\ A_{\boldsymbol{\alpha}}(vA_{\boldsymbol{\beta}-\boldsymbol{\varepsilon}_i})\subset v(A^*_{\boldsymbol{\alpha}}A_{\boldsymbol{\beta}-\boldsymbol{\varepsilon}_i})=vA_{\boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\varepsilon}_i}=A'_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \\ \quad \text{if } \boldsymbol{\alpha}\in\Gamma_i,\ \boldsymbol{\beta}\notin\Gamma_i, \\ (vA_{\boldsymbol{\beta}})A_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_i}\subset v(A_{\boldsymbol{\beta}}A_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_i})=vA_{\boldsymbol{\beta}+\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_i}=A'_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \\ \quad \text{if } \boldsymbol{\alpha}\notin\Gamma_i,\ \boldsymbol{\beta}\in\Gamma_i, \\ (vA_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_i})(vA_{\boldsymbol{\beta}-\boldsymbol{\varepsilon}_i})\subset zA^*_{\boldsymbol{\beta}-\boldsymbol{\varepsilon}_i}A_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_i}=A'_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \\ \quad \text{if } \boldsymbol{\alpha}\notin\Gamma_i,\ \boldsymbol{\beta}\notin\Gamma_i, \end{cases}$$

since $*$ is graded and $z\in A_{2\boldsymbol{\varepsilon}_i}$. Hence $(A,z)=\bigoplus_{\boldsymbol{\alpha}\in\Lambda}A'_{\boldsymbol{\alpha}}$ is a Λ-graded algebra over F. Since $\{\boldsymbol{\varepsilon}_1,\ldots,\boldsymbol{\varepsilon}_n\}$ is a basis of Λ, the uniqueness of Λ-grading is clear. Also by Lemma 3.5, (A,z) is an alternative algebra. Moreover, we know that $0\neq x\in A_{\boldsymbol{\alpha}}$ is invertible. Since z is invertible, v is invertible in the alternative algebra (A,z) (see Remark 3.3(2)). Hence, vx is invertible (see [1, Lemma 9, p. 205]). Consequently, (A,z) is a division Λ-graded alternative algebra over F. Finally, by (3.2), it is clear that $*$ on (A,z) is graded. □

Corollary 5.4. *Let $\{\boldsymbol{\varepsilon}_1,\ldots,\boldsymbol{\varepsilon}_n\}$ be a basis of $\mathbb{Z}^n$ and*

$$\Gamma^{(m)}:=2\mathbb{Z}\boldsymbol{\varepsilon}_1+\cdots+2\mathbb{Z}\boldsymbol{\varepsilon}_m+\mathbb{Z}\boldsymbol{\varepsilon}_{m+1}+\cdots+\mathbb{Z}\boldsymbol{\varepsilon}_n\quad \textit{for } m=1,2,3.$$

Let $A=\bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(m)}}A_{\boldsymbol{\alpha}}$ be a division $\Gamma^{(m)}$-graded associative algebra over a field K and R a subalgebra contained in the centre of A. Suppose

(i) *A has a scalar involution $*$ over R which is graded*;
(ii) *there exist $z_i\in A_{2\boldsymbol{\varepsilon}_i}\cap R$ for all $1\leqslant i\leqslant m$ such that each z_i is invertible in R*;
(iii) *A is commutative if $m=2$, and A is commutative and $*$ is trivial if $m=3$.*

Then the algebra $(A,z_1,\ldots,z_m)$ obtained by the Cayley–Dickson process over R has a unique $\mathbb{Z}^n$-grading such that every homogeneous space of degree $\boldsymbol{\alpha}\in\Gamma^{(m)}$ in $(A,z_1,\ldots,z_m)$ is equal to $A_{\boldsymbol{\alpha}}$ and each $v_i=(0,1)\in((A,z_1,\ldots,z_{i-1}),z_i)$ has degree $\boldsymbol{\varepsilon}_i$. Also by the $\mathbb{Z}^n$-grading, $(A,z_1,\ldots,z_m)$ becomes a division $\mathbb{Z}^n$-graded alternative algebra over K.

Moreover, assume that

(iv) *A is not commutative if $m=1$, $*$ is not trivial if $m=2$, and $\operatorname{ch}K\neq 2$ if $m=3$.*

Then $(A,z_1,\ldots,z_m)$ is not associative.

Proof. For $m=1$, we take $\Lambda=\mathbb{Z}^n$ in Lemma 5.3; and so $\Gamma^{(1)}=\Gamma_1$. Then we take $z=z_1$ in Lemma 5.3 and get the required division $\mathbb{Z}^n$-graded alternative algebra (A,z_1) over K. Moreover, (iv) and Lemma 3.5 implies that (A,z_1) is not associative.

To show the cases $m=2$ and $m=3$, we define the following: for $j=1,2$, let

$$\Gamma_j^{(m)} := \mathbb{Z}\boldsymbol{\varepsilon}_1+\cdots+\mathbb{Z}\boldsymbol{\varepsilon}_j+2\mathbb{Z}\boldsymbol{\varepsilon}_{j+1}+\cdots+2\mathbb{Z}\boldsymbol{\varepsilon}_m+\mathbb{Z}\boldsymbol{\varepsilon}_{m+1}+\cdots+\mathbb{Z}\boldsymbol{\varepsilon}_n.$$

For $m=2$, we take $\Lambda=\Gamma_1^{(2)}$, $\Gamma_i=\Gamma^{(2)}$, and $z=z_1$ in Lemma 5.3. Then, by (iii), we get the division $\Gamma_1^{(2)}$-graded associative algebra (A,z_1) which has the graded scalar involution $*$ over R. Note that R is contained in the centre of (A,z_1) and that $z_2\in A_{2\boldsymbol{\varepsilon}_2}\cap R=(A,z_1)_{2\boldsymbol{\varepsilon}_2}\cap R$ is invertible in R. Thus one can apply Lemma 5.3 again for $\Lambda=\mathbb{Z}^n$, $\Gamma_i=\Gamma_1^{(2)}$ and $z=z_2$, and get the required division $\mathbb{Z}^n$-graded alternative algebra (A,z_1,z_2) over K. Moreover, (iv) implies that (A,z_1) is not commutative and that (A,z_1,z_2) is not associative.

For $m=3$, by the same way as the case $m=2$, one can apply Lemma 5.3 three times. Namely:

1. Take $\Lambda=\Gamma_1^{(3)}$, $\Gamma_i=\Gamma^{(3)}$, and $z=z_1$, and get the division $\Gamma_1^{(3)}$-graded commutative associative algebra (A,z_1).
2. Take $\Lambda=\Gamma_2^{(3)}$, $\Gamma_i=\Gamma_1^{(3)}$, and $z=z_2$, and get the division $\Gamma_2^{(3)}$-graded associative algebra (A,z_1,z_2).
3. Take $\Lambda=\mathbb{Z}^n$, $\Gamma_i=\Gamma_2^{(3)}$, and $z=z_3$, and get the required division $\mathbb{Z}^n$-graded alternative algebra (A,z_1,z_2,z_3) over K.

Moreover, (iv) implies that (A,z_1) has the non-trivial graded scalar involution, that (A,z_1,z_2) is not commutative and that (A,z_1,z_2,z_3) is not associative. □

We are now ready to construct four octonion rings.

Construction 5.5. Let K be a field extension of F and let $\{\boldsymbol{\varepsilon}_1,\dots,\boldsymbol{\varepsilon}_n\}$ be a basis of $\mathbb{Z}^n$.

(1) Let $\mathbb{O}$ be an octonion division algebra over K and

$$K\left[z_1^{\pm1},\dots,z_n^{\pm1}\right]=\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}Kz_{\boldsymbol{\alpha}}$$

the algebra of Laurent polynomials over K where $z_{\boldsymbol{\alpha}}=z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ for $\boldsymbol{\alpha}=\alpha_1\boldsymbol{\varepsilon}_1+\cdots+\alpha_n\boldsymbol{\varepsilon}_n\in\mathbb{Z}^n$. Then, by Lemma 2.2,

$$\mathbb{O}_1:=\mathbb{O}\otimes_K K\left[z_1^{\pm1},\dots,z_n^{\pm1}\right]=\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}(\mathbb{O}\otimes_K Kz_{\boldsymbol{\alpha}})$$

is a division $\mathbb{Z}^n$-graded alternative algebra over K and hence over F. It is clearly alternative but not associative.

(2) Let $\mathbb{H}$ be a quaternion division algebra over K with scalar involution $*$. Let

$$\Gamma^{(1)} := 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + \mathbb{Z}\boldsymbol{\varepsilon}_2 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n,$$
$$R := K\big[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\big] = \bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(1)}} K z_{\boldsymbol{\alpha}},$$

where $z_{\boldsymbol{\alpha}} = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ for $\boldsymbol{\alpha} = 2\alpha_1\boldsymbol{\varepsilon}_1 + \alpha_2\boldsymbol{\varepsilon}_2 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \Gamma^{(1)}$. Then, by Lemma 2.2,

$$\mathbb{H}_1 := \mathbb{H} \otimes_K R = \bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(1)}} (\mathbb{H} \otimes_K K z_{\boldsymbol{\alpha}})$$

is a division $\Gamma^{(1)}$-graded associative algebra over F and it is clearly associative but not commutative. One can check that $\mathbb{H}_1$ has the scalar involution $*$ over R defined by $(x \otimes z)^* = x^* \otimes z$ for $x \in \mathbb{H}$, $z \in R$; and so $*$ is clearly graded on $\mathbb{H}_1$. Let $Z(\mathbb{H}_1)$ be the centre of $\mathbb{H}_1$. Then, $R \subset Z(\mathbb{H}_1)$ (in fact they are equal), $z_1 \in R$ has degree $2\boldsymbol{\varepsilon}_1$ in $\mathbb{H}_1$ and z_1 is invertible in R. Thus we can apply Corollary 5.4. Namely, the algebra

$$\mathbb{O}_2 := (\mathbb{H}_1, z_1)$$

obtained by the Cayley–Dickson process over R is a division $\mathbb{Z}^n$-graded alternative algebra over K, which is not associative. For $x \in \mathbb{H}$ and $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_2 = (\mathbb{H}_1, z_1)$:

$$x t_{\boldsymbol{\alpha}} := \begin{cases} (x \otimes z_{\boldsymbol{\alpha}}, 0), & \text{if } \alpha_1 \equiv 0 \bmod 2, \\ (0, x \otimes z_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_1}), & \text{if } \alpha_1 \equiv 1 \bmod 2. \end{cases}$$

Note that $t_{\boldsymbol{\varepsilon}_1} = (0, 1)$ and $t_{\boldsymbol{\varepsilon}_1}^2 = z_1$. One can write $\mathbb{O}_2 = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} \mathbb{H} t_{\boldsymbol{\alpha}}$.

(3) Assume that $n \geqslant 2$. Let $\mathbb{E}$ be a separable quadratic field extension of K. Hence the non-trivial Galois automorphism over K is a scalar involution denoted by $*$. Let

$$\Gamma^{(2)} := 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + 2\mathbb{Z}\boldsymbol{\varepsilon}_2 + \mathbb{Z}\boldsymbol{\varepsilon}_3 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n,$$
$$R := K\big[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\big] = \bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(2)}} K z_{\boldsymbol{\alpha}},$$

where $z_{\boldsymbol{\alpha}} = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ for $\boldsymbol{\alpha} = 2\alpha_1\boldsymbol{\varepsilon}_1 + 2\alpha_2\boldsymbol{\varepsilon}_2 + \alpha_3\boldsymbol{\varepsilon}_3 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \Gamma^{(2)}$. Then,

$$\mathbb{E}_1 := \mathbb{E} \otimes_K R = \bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(2)}} (\mathbb{E} \otimes_K K z_{\boldsymbol{\alpha}})$$

is a division $\Gamma^{(2)}$-graded commutative associative algebra over K and has the graded scalar involution $*$ over R by the same argument as used in (2). We have $R \subset Z(\mathbb{E}_1) = \mathbb{E}_1$, Also, z_1 and z_2 satisfy the condition in Corollary 5.4. Thus we get a division $\mathbb{Z}^n$-graded alternative algebra

$$\mathbb{O}_3 := (\mathbb{E}_1, z_1, z_2)$$

over K, which is not associative. For $x \in \mathbb{E}$ and $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \alpha_2\boldsymbol{\varepsilon}_2 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_3 = (\mathbb{E}_1, z_1, z_2)$:

$$t_{\boldsymbol{\alpha}} := \begin{cases} \big((z_{\boldsymbol{\alpha}}, 0), (0,0)\big), & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \bmod 2, \\ \big((0, z_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_1}), (0,0)\big), & \text{if } \alpha_1 \equiv 1 \text{ and } \alpha_2 \equiv 0 \bmod 2, \\ \big((0,0), (z_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_2}, 0)\big), & \text{if } \alpha_1 \equiv 0 \text{ and } \alpha_2 \equiv 1 \bmod 2, \\ \big((0,0), (0, z_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2})\big), & \text{if } \alpha_1 \equiv 1 \text{ and } \alpha_2 \equiv 1 \bmod 2. \end{cases}$$

Note that $t_{\boldsymbol{\varepsilon}_1} = ((0,1),(0,0))$, $t_{\boldsymbol{\varepsilon}_2} = ((0,0),(1,0))$, $t_{\boldsymbol{\varepsilon}_1}^2 = z_1$ and $t_{\boldsymbol{\varepsilon}_2}^2 = z_2$. One can write $\mathbb{O}_3 = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} \mathbb{E}t_{\boldsymbol{\alpha}}$.

(4) Assume that $n \geqslant 3$ and that $\operatorname{ch} F \neq 2$. We can apply Corollary 5.4 for

$$\begin{aligned} \Gamma^{(3)} &:= 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + 2\mathbb{Z}\boldsymbol{\varepsilon}_2 + 2\mathbb{Z}\boldsymbol{\varepsilon}_3 + \mathbb{Z}\boldsymbol{\varepsilon}_4 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n, \\ A = R &:= K\big[z_1^{\pm1}, \ldots, z_n^{\pm1}\big] = \bigoplus_{\boldsymbol{\alpha}\in\Gamma^{(3)}} K z_{\boldsymbol{\alpha}}, \end{aligned}$$

where $z_{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\boldsymbol{\alpha} = 2\alpha_1\boldsymbol{\varepsilon}_1 + 2\alpha_2\boldsymbol{\varepsilon}_2 + 2\alpha_3\boldsymbol{\varepsilon}_3 + \alpha_4\boldsymbol{\varepsilon}_4 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \Gamma^{(3)}$. In fact, A has the trivial scalar involution, i.e., the identity map, and z_1, z_2, z_3 satisfy the condition in Corollary 5.4. Thus we get a division $\mathbb{Z}^n$-graded alternative algebra

$$\mathbb{O}_4 := (R, z_1, z_2, z_3)$$

over K, which is not associative. It is clear that every homogeneous space is 1-dimensional over K; and so $\mathbb{O}_4$ is an alternative torus over K. It is called the *octonion torus* or the *Cayley torus* over K. This torus was first discovered in [3]. We present a slightly different description of the octonion torus. Consider the algebra

$$\big(K\big[z_1^{\pm1}, \ldots, z_n^{\pm1}\big], z_1, z_2\big),$$

which is associative (so a quantum torus over K). One can easily check that this algebra is graded isomorphic to a quaternion torus $K_{\boldsymbol{h}} = K_{\boldsymbol{h}}[u_1^{\pm1}, \ldots, u_n^{\pm1}]$ as defined in Example 4.2, via

$$\begin{aligned} &\big((0,1),(0,0)\big) \leftrightarrow u_1, \qquad \big((0,0),(1,0)\big) \leftrightarrow u_2, \\ &z_3 \leftrightarrow u_3, \quad \ldots, \quad z_n \leftrightarrow u_n. \end{aligned}$$

We identify them. So the octonion torus over K is

$$\mathbb{O}_4 = (K_{\boldsymbol{h}}, u_3)$$

and the $\mathbb{Z}^n$-grading comes from the following Γ_3-grading of $K_{\boldsymbol{h}}$ (see Lemma 5.3):

$$\begin{aligned} \Gamma_3 &= \mathbb{Z}\boldsymbol{\varepsilon}_1 + \mathbb{Z}\boldsymbol{\varepsilon}_2 + 2\mathbb{Z}\boldsymbol{\varepsilon}_3 + \mathbb{Z}\boldsymbol{\varepsilon}_4 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n, \\ K_{\boldsymbol{h}} &= \bigoplus_{\boldsymbol{\alpha}\in\Gamma_3} K u_{\boldsymbol{\alpha}}, \end{aligned}$$

where $u_{\boldsymbol{\alpha}} = u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \cdots u_n^{\alpha_n}$ for $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \alpha_2\boldsymbol{\varepsilon}_2 + 2\alpha_3\boldsymbol{\varepsilon}_3 + \alpha_4\boldsymbol{\varepsilon}_4 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n$. For $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_4 = (K_{\boldsymbol{h}}, u_3)$:

$$t_{\boldsymbol{\alpha}} := \begin{cases} (u_{\boldsymbol{\alpha}}, 0), & \text{if } \alpha_3 \equiv 0 \bmod 2, \\ (0, u_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_3}), & \text{if } \alpha_3 \equiv 1 \bmod 2. \end{cases}$$

Then we have $\mathbb{O}_4 = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} K t_{\boldsymbol{\alpha}}$ as K-spaces. Note that $t_{\boldsymbol{\varepsilon}_3} = (0, 1)$ and $t_{\boldsymbol{\varepsilon}_3}^2 = u_3$. Note that the centre of $\mathbb{O}_4$ is

$$K\left[u_1^{\pm 2}, u_2^{\pm 2}, u_3^{\pm 1}, \ldots, u_n^{\pm 1}\right] = K\left[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\right].$$

Also, the structure constants of $\mathbb{O}_4$ relative to $\{t_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathbb{Z}^n}$ are 1 or -1.

All the gradings of $\mathbb{O}_1$, $\mathbb{O}_2$, $\mathbb{O}_3$, and $\mathbb{O}_4$ determined by a basis $\langle \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n \rangle$ of $\mathbb{Z}^n$ described in Construction 5.5 are called *toral gradings*.

Notice that $\mathbb{O}_1$, $\mathbb{H}_1$, and $\mathbb{E}_1$ in Construction 5.5 are also the algebras obtained by the Cayley–Dickson process over $K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. More generally, we have the following lemma.

Lemma 5.6. *Let R be a commutative associative algebra over a field K and $\mu_1, \ldots, \mu_l \in K$. Then,*

$$(K, \mu_1, \ldots, \mu_l) \otimes_K R \cong (R, \mu_1, \ldots, \mu_l) \quad \textit{as } R\textit{-algebras.}$$

Proof. Straightforward. □

Now we prove our main result.

Theorem 5.7. *A division $\mathbb{Z}^n$-graded alternative algebra $A = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} A_{\boldsymbol{\alpha}}$ over F which is not associative is graded isomorphic over $K = Z_{\mathbf{0}}$ to one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and $\mathbb{O}_4$ for some toral gradings. In other words, when* $\operatorname{ch} F \neq 2$, *$A$ is isomorphic over K to $(K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}], \mu_1, \mu_2, \mu_3)$ where*

(1) $0 \neq \mu_1, \mu_2, \mu_3 \in K$ *such that (K, μ_1, μ_2, μ_3) is an octonion division algebra over K,*
(2) $0 \neq \mu_1, \mu_2 \in K$, *and $\mu_3 = z_1$ such that (K, μ_1, μ_2) is a quaternion division algebra over K,*
(3) $0 \neq \mu_1 \in K$, *$\mu_2 = z_1$, and $\mu_3 = z_2$ such that (K, μ_1) is a quadratic field extension of K, or*
(4) $\mu_1 = z_1$, $\mu_2 = z_2$, *and* $\mu_3 = z_3$,

and (1), (2), *or* (3) *when* $\operatorname{ch} F = 2$.

Also, these four algebras are all non-isomorphic.

Proof. We already know that A is an octonion ring whose central closure $\overline{A}$ is an octonion division algebra over $\overline{Z}$. By Lemma 2.4, $\overline{A}$ is a $\mathbb{Z}^n/\Gamma$-graded alternative algebra, where $\Gamma = \{\boldsymbol{\alpha} \in \mathbb{Z}^n \mid Z_{\boldsymbol{\alpha}} = Z \cap A_{\boldsymbol{\alpha}} \neq 0\}$ is the central grading group of A; i.e.,

$$\overline{A} = \bigoplus_{\bar{\boldsymbol{\alpha}} \in \mathbb{Z}^n/\Gamma} \overline{A}_{\bar{\boldsymbol{\alpha}}} \quad \text{where } \overline{A}_{\bar{\boldsymbol{\alpha}}} = \overline{Z} \otimes_Z A_{\bar{\boldsymbol{\alpha}}} \text{ and } A_{\bar{\boldsymbol{\alpha}}} = ZA_{\boldsymbol{\alpha}}.$$

Since $\overline{A}$ is a division algebra, it is, in particular, a division $\mathbb{Z}^n/\Gamma$-graded algebra of dimension 8. Hence every homogeneous space has the same dimension. Also, since the octonion algebra $\overline{A}$ has a scalar involution, $\mathbb{Z}^n/\Gamma$ is either (0) or an elementary group of exponent 2, by Lemma 4.1. Therefore, we have four cases; there exists a basis $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$ of $\mathbb{Z}^n$ such that for $m = 0, 1, 2, 3$,

$$\Gamma = 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + \cdots + 2\mathbb{Z}\boldsymbol{\varepsilon}_m + \mathbb{Z}\boldsymbol{\varepsilon}_{m+1} + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n.$$

Let

$$0 \neq t_i \in A_{\boldsymbol{\varepsilon}_i} \quad \text{for } 1 \leqslant i \leqslant m.$$

By Lemma 2.3(iii), we have $A = \bigoplus_{\bar{\boldsymbol{\alpha}} \in \mathbb{Z}^n/\Gamma} A_{\bar{\boldsymbol{\alpha}}}$; and hence

$$A = \langle ZA_{\mathbf{0}}, t_i \rangle_{1 \leqslant i \leqslant m}$$

as a Z-algebra. Also by Lemma 4.1(iii), we have

$$n(t_i, ZA_{\mathbf{0}}) = 0.$$

Moreover, by Lemma 4.1(ii), $t_i^2 \in \overline{Z}1$; and so $t_i^2 \in \overline{Z}1 \cap A_{2\boldsymbol{\varepsilon}_i} = Z(\overline{A}) \cap A_{2\boldsymbol{\varepsilon}_i} = Z_{2\boldsymbol{\varepsilon}_i}$. Choose $0 \neq z_j \in Z_{\boldsymbol{\varepsilon}_j}$ for $m < j \leqslant n$ and put $z_i := t_i^2$. Since Z is a commutative associative Γ-torus over K, one can write

$$Z = K\bigl[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\bigr] = \bigoplus_{\boldsymbol{\alpha} \in \Gamma} Kz_{\boldsymbol{\alpha}},$$

where $z_{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\boldsymbol{\alpha} = 2\alpha_1\boldsymbol{\varepsilon}_1 + \cdots + 2\alpha_m\boldsymbol{\varepsilon}_m + \alpha_{m+1}\boldsymbol{\varepsilon}_{m+1} + \cdots + \alpha_n\boldsymbol{\varepsilon}_n \in \Gamma$.

Case (I). $m = 0$, i.e., $\Gamma = \mathbb{Z}^n$.

By Lemma 2.4(ii), we have $\dim_K A_{\mathbf{0}} = \dim_{\overline{Z}} \overline{A}_{\bar{\mathbf{0}}} = \dim_{\overline{Z}} \overline{A} = 8$. Since $A_{\mathbf{0}}$ is an alternative division algebra over K, it is strongly prime, in particular. Since A is not associative, neither is $A_{\mathbf{0}}$. Hence, $A_{\mathbf{0}}$ is an octonion algebra over its centre, by Slater's Theorem (5.1). However, since $\dim_K A_{\mathbf{0}} = 8$, $A_{\mathbf{0}}$ is already central over K and is an octonion division algebra over K. By Lemma 2.3, we can identify (and always do below)

$$ZA_{\mathbf{0}} = A_{\mathbf{0}} \otimes_K Z.$$

Hence we get $A = ZA_{\mathbf{0}} = A_{\mathbf{0}} \otimes_K K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] = \mathbb{O}_1$ in Construction 5.5(1).

For the next two cases, we prove a lemma. Recall that A is a subalgebra of $\overline{A}$ over Z, by identifying $x = 1 \otimes x \in \overline{A}$ for $x \in A$.

Lemma 5.8. *Let $*$ be the scalar involution of the octonion division $\overline{Z}$-algebra $\overline{A}$. Suppose that the K-algebra $A_{\mathbf{0}}$ has a scalar involution $\bar{\ }$. Then, $\sigma := * |_{ZA_{\mathbf{0}}}$ is a scalar involution of Z-algebra $ZA_{\mathbf{0}}$; and for $z_i \in Z$, $x_i \in A_{\mathbf{0}}$, we have $\sigma(\sum_i z_i x_i) = \sum_i z_i \overline{x_i}$.*

Proof. By Lemma 2.4(ii), one can easily check that

$$\overline{A}_{\bar{\mathbf{0}}} = \overline{Z} \otimes_Z A_{\bar{\mathbf{0}}} \cong \overline{Z} \otimes_K A_{\mathbf{0}}$$

over $\overline{Z}$ via $y \otimes x \mapsto y \otimes x$ for $y \in \overline{Z}$ and $x \in A_{\mathbf{0}}$. We can naturally extend $\bar{\ }$ to the subalgebra $\overline{A}_{\bar{\mathbf{0}}}$ of $\overline{A}$ over $\overline{Z}$. Namely, we can define $\overline{y \otimes x} := y \otimes \bar{x}$, which is still a scalar involution over $\overline{Z}$. But $\overline{A}_{\bar{\mathbf{0}}}$ has another scalar involution, that is, $* := * |_{\overline{A}_{\bar{\mathbf{0}}}}$. So we get $\bar{\ } = *$ on $\overline{A}_{\bar{\mathbf{0}}}$ since a scalar involution over a field is unique by Lemma 3.7. Finally, one can check that σ is a scalar involution of $ZA_{\mathbf{0}}$ over Z. $\square$

We denote this σ also by $*$.

Case (II). $m = 1$, i.e., $\Gamma = 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + \mathbb{Z}\boldsymbol{\varepsilon}_2 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\overline{Z}} \overline{A}_{\bar{\mathbf{0}}} = 4$. Then, by Lemma 3.8, $\overline{A}_{\bar{\mathbf{0}}} = (\overline{A}_{\bar{\mathbf{0}}}, *)$ is a quaternion division algebra over $\overline{Z}$ with the restricted scalar involution $*$ and $\overline{A} = (\overline{A}_{\bar{\mathbf{0}}}, \mu)$ for some structure constant $0 \neq \mu \in \overline{Z}$. Also by Lemma 2.4(ii), $A_{\mathbf{0}}$ is a 4-dimensional associative non-commutative division algebra over K. Hence it has to be central and has degree 2. It is well known that a central simple associative algebra of degree 2 over a field is a quaternion algebra (see, e.g., [12]). Hence, $A_{\mathbf{0}}$ has a scalar involution over K and, by Lemma 5.8, $ZA_{\mathbf{0}}$ has the scalar involution $* = * |_{ZA_{\mathbf{0}}}$ over Z.

We show the following lemma.

Lemma 5.9. *Let $B = (B, *)$ be an associative composition algebra over a field L. Let $C = (B, \mu)$ for $0 \neq \mu \in L$ and R a subring of L. Suppose*:

- (i) *There exists $t \in C$ such that $n(t, B) = 0$, where n is the norm on C.*
- (ii) $0 \neq t^2 \in R1 = R$ (*we identify $L1$ with L*).
- (iii) *P is a subalgebra of B over R such that the restriction of $*$ to P is a scalar involution over R.*

Then the algebra generated by P and t over R is equal to the algebra obtained by the Cayley–Dickson process with structure constant t^2 over R:

$$\langle P, t\rangle = \left(P, t^2\right) \quad \textit{and} \quad t = (0, 1) \quad \textit{in } \left(P, t^2\right).$$

Moreover, the restriction of the scalar involution $*$ of C to (P, t^2) is a scalar involution over R.

Proof. Since B is a composition algebra, the norm $n(\cdot\,,\cdot)$ on C is non-degenerate and the dimension of C over L is finite. From a well-known theorem in linear algebra, we have $\dim B + \dim B^{\perp} = \dim C$. By (3.6), we have $vB \subset B^{\perp}$. Since $\dim B + \dim vB = \dim C$, we get $vB = B^{\perp}$. Thus we have $t = ve$ for some $0 \neq e \in B$, by (i). Also, by (ii) we have $0 \neq t^2 = (ve)^2 = \mu n(e)1 \in R$, which is cancellable. Now, we have by (3.1),

$$tP = (ve)P = v(Pe) \subset vB;$$

and hence, $\langle P, t\rangle = P \oplus tP$. By (iii), P has a scalar involution $* = *\,|_P$ over R. We have

$$(a+tb)(c+td) = \left(ac + t^2 db^*\right) + t(a^*d + cb) \quad \text{for all } a, b, c, d \in P.$$

In fact, one checks, using $t = ve$, (3.1), and the associativity of P: $a(td) = t(a^*d)$, $(tb)c = t(cb)$, and $(tb)(td) = t^2 db^*$. Hence $\langle P, t\rangle = P \oplus tP = (P, t^2)$ and $t = (0, 1)$ in (P, t^2). Moreover, for $b \in P$,

$$\begin{aligned}(tb)^* &= b^*(ve)^* = b^*(e^*v^*) = b^*(-e^*v) = -b^*(ve) = -v(be)\\ &= -(ve)b = -tb.\end{aligned}$$

So, $*\,|_{(P,t^2)}$ coincides with the scalar involution of (P, t^2) over R. $\square$

We apply Lemma 5.9 for $A = \langle ZA_{\mathbf{0}}, t_1\rangle$:

$$\begin{aligned}&L := \overline{Z}, \qquad R := Z, \qquad B := \left(\overline{A}_{\bar{\mathbf{0}}}, *\right),\\ &P := ZA_{\mathbf{0}} \subset B, \quad \text{having the scalar involution } * \text{ over } R,\\ &C := \overline{A} = (B, \mu) \quad \text{and} \quad t := t_1.\end{aligned}$$

So we get

$$A = \langle ZA_{\mathbf{0}}, t_1\rangle = (ZA_{\mathbf{0}}, z_1) \quad \text{and} \quad t_1 = (0, 1).$$

Let $\mathbb{H} := A_{\mathbf{0}}$ which is the quaternion division algebra over K described above, in which case the division Γ-graded associative algebra

$$ZA_{\mathbf{0}} = \mathbb{H} \otimes_K K\left[z_1^{\pm 1}, \ldots, z_n^{\pm 1}\right]$$

is equal to $\mathbb{H}_1$ constructed in Construction 5.5(2). Thus we get

$$A = (ZA_{\mathbf{0}}, z_1) = (\mathbb{H}_1, z_1) = \mathbb{O}_2 \quad \text{in Construction 5.5(2)},$$

which have the same $\mathbb{Z}^n$-gradings (see the uniqueness part of $\mathbb{Z}^n$-gradings in Lemma 5.3).

Case (III). $m = 2$, i.e., $\Gamma = 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + 2\mathbb{Z}\boldsymbol{\varepsilon}_2 + \mathbb{Z}\boldsymbol{\varepsilon}_3 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\overline{Z}} \overline{A}_{\bar{\mathbf{0}}} = 2$. Thus, $\overline{A}_{\bar{\mathbf{0}}} = (\overline{A}_{\bar{\mathbf{0}}}, *)$ is a quadratic field extension of $\overline{Z}$ with the restricted scalar involution $*$ and, by Lemma 3.8, through the 4-dimensional subalgebra $B_1 := \overline{A}_{\bar{\mathbf{0}}} \oplus \overline{A}_{\bar{\varepsilon}_1}$,

$$\overline{A} = \left(\overline{A}_{\bar{\mathbf{0}}}, \mu_1, \mu_2\right)$$

for some structure constants $0 \neq \mu_1, \mu_2 \in \overline{Z}$. Also by Lemma 2.4(ii), $A_{\mathbf{0}}$ is a quadratic field extension of K. Hence, $A_{\mathbf{0}}$ has a scalar involution over K and, by Lemma 5.8, $ZA_{\mathbf{0}}$ has the scalar involution $* = * \,|_{ZA_{\mathbf{0}}}$ over Z. Also, if $A_{\mathbf{0}}$ is not separable, then the scalar involution of $A_{\mathbf{0}}$ is trivial. So, $*$ becomes trivial and $\overline{A}$ becomes commutative and associative. This contradicts our setting. Therefore, $A_{\mathbf{0}}$ is a separable quadratic field extension of K.

We show the following lemma which is a corollary of Lemma 5.9.

Lemma 5.10. *Let $B = (B, *)$ be an associative composition algebra over a field L and R a subring of L. Let $C = (B, \mu_1, \ldots, \mu_m)$ for $0 \neq \mu_i \in L$, $1 \leqslant m \leqslant 3$, be an alternative algebra over L. Let $B_0 := B$ and $B_i := (B_{i-1}, \mu_i)$, $1 \leqslant i \leqslant m$. (All B_i have the restricted scalar involution $*$ of C.) Suppose*:

(i) *There exist $t_1, \ldots, t_m \in C$ such that $n(t_i, B_{i-1}) = 0$.*
(ii) $0 \neq t_i^2 \in R1 = R$ *for all* $1 \leqslant i \leqslant m$.
(iii) *P is a subalgebra of B over R such that the restriction of $*$ to P is a scalar involution over R.*

Then

$$\langle P, t_1, \ldots, t_m\rangle = \left(P, t_1^2, \ldots, t_m^2\right)$$
$$\text{and} \quad \text{each } t_i = (0, 1) \quad \text{in } \left(\left(P, t_1^2, \ldots, t_{i-1}^2\right), t_i^2\right).$$

Proof. The case $m = 1$ is done by Lemma 5.9. For $m > 1$, assume that the subalgebra P_{m-1} of B_{m-1} generated by $P, t_1, \ldots, t_{m-1}$ over R is equal to $(P, t_1^2, \ldots, t_{m-1}^2)$ and that the restriction of the scalar involution $*$ of B_{m-1} to $(P, t_1^2, \ldots, t_{m-1}^2)$ is a scalar involution over R. Then, since $C = (B_{m-1}, \mu_m)$ has the element t_m satisfying $n(t_m, B_{m-1}) = 0$ and $t_m^2 \in R$, we can apply Lemma 5.9 for $B = B_{m-1}$, $P = P_{m-1}$ and $t = t_m$. Namely,

$$\langle P, t_1, \ldots, t_m\rangle = \langle P_{m-1}, t_m\rangle = \left(P_{m-1}, t_m^2\right) = \left(P, t_1^2, \ldots, t_m^2\right),$$

and each t_i has the required form. □

As in Case (II), we apply Lemma 5.10 and get

$$A = \langle ZA_{\mathbf{0}}, t_1, t_2\rangle = (ZA_{\mathbf{0}}, z_1, z_2),$$

$t_1 = ((0,1),(0,0))$, and $t_2 = ((0,0),(1,0))$. Let $\mathbb{E} := A_{\mathbf{0}}$, which is the separable quadratic field extension of K shown above, in which case the division Γ-graded commutative associative algebra

$$ZA_{\mathbf{0}} = \mathbb{E} \otimes_K K\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]$$

is equal to $\mathbb{E}_1$ constructed in Construction 5.5(3). Thus we get

$$A = (ZA_{\mathbf{0}}, z_1, z_2) = (\mathbb{E}_1, z_1, z_2) = \mathbb{O}_3 \quad \text{in Construction 5.5(3),}$$

which have the same $\mathbb{Z}^n$-gradings.

Case (IV). $m = 3$, i.e., $\Gamma = 2\mathbb{Z}\boldsymbol{\varepsilon}_1 + 2\mathbb{Z}\boldsymbol{\varepsilon}_2 + 2\mathbb{Z}\boldsymbol{\varepsilon}_3 + \mathbb{Z}\boldsymbol{\varepsilon}_4 + \cdots + \mathbb{Z}\boldsymbol{\varepsilon}_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\overline{Z}} \overline{A}_{\bar{\mathbf{0}}} = 1$. Thus we have $\overline{Z}1 = \overline{A}_{\bar{\mathbf{0}}} = (\overline{A}_{\bar{\mathbf{0}}}, *)$ with the restricted scalar involution $*$ which is the trivial involution and, by Lemma 3.8, through the subalgebras $B_2 = \overline{A}_{\bar{\mathbf{0}}} \oplus \overline{A}_{\bar{\boldsymbol{\varepsilon}}_1} \oplus \overline{A}_{\bar{\boldsymbol{\varepsilon}}_2} \oplus \overline{A}_{\overline{\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_2}}$, and B_1,

$$\overline{A} = \left(\overline{A}_{\bar{\mathbf{0}}}, \mu_1, \mu_2, \mu_3\right),$$

for some structure constants $0 \neq \mu_1, \mu_2, \mu_3 \in \overline{Z}$ if $\operatorname{ch} F \neq 2$.

Note that $A_{\mathbf{0}} = K$ and $ZA_{\mathbf{0}} = Z$ in this case. As in Case (II), we apply Lemma 5.10 and get

$$A = \langle Z, t_1, t_2, t_3 \rangle = (Z, z_1, z_2, z_3) = \mathbb{O}_4 \quad \text{in Construction 5.5(4),}$$

which have the same $\mathbb{Z}^n$-gradings. If $\operatorname{ch} F = 2$, then $B_1 = \overline{A}_{\bar{\mathbf{0}}} \oplus \overline{A}_{\bar{\boldsymbol{\varepsilon}}_1}$ is a purely inseparable extension field of $\overline{A}_{\bar{\mathbf{0}}}$ by Lemma 4.1(i). Hence, B_1 is not a composition algebra, which contradicts Lemma 3.8. Therefore, Case (IV) cannot happen if $\operatorname{ch} F = 2$.

The different descriptions of the octonion rings follow from Lemma 5.6. The last statement follows from the fact that $A_{\mathbf{0}}$ is an isomorphic invariant (which is easily shown). □

Example 5.11. We choose our base field F to be $\mathbb{R}$, the field of real numbers. Then there exist a unique quadratic field extension $\mathbb{C}$, the field of complex numbers, a unique quaternion division algebra $\mathbb{H}$, Hamilton's quaternion, and a unique octonion division algebra $\mathbb{O}$, the algebras of Cayley numbers. Hence there exist four division $\mathbb{Z}^n$-graded alternative but not associative algebras over $\mathbb{R}$, namely, taking K to be $\mathbb{R}$ in Construction 5.5, $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and $\mathbb{O}_4$, the Cayley torus over $\mathbb{R}$.

Moreover, if we assume that a homogeneous space is finite-dimensional (and hence all homogeneous spaces are finite-dimensional), then any division $\mathbb{Z}^n$-graded alternative but not associative algebra over $\mathbb{R}$ is isomorphic to one of the four algebras above or another $\mathbb{O}_4$, the Cayley torus over $\mathbb{C}$, taking K to be $\mathbb{C}$ in Construction 5.5. Indeed, we know that finite-dimensional field extension K

of $\mathbb{R}$ is $\mathbb{R}$ or $\mathbb{C}$. If $K=\mathbb{R}$, then we get the four algebras above. If $K=\mathbb{C}$, then the only finite-dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$. Hence we have $A_{\mathbf{0}}=K=\mathbb{C}$ and only Case (IV) appears in Theorem 5.7.

We showed that any division $\mathbb{Z}^n$-graded associative algebra is isomorphic to $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ which is a natural generalization of quantum tori (see [2]).

Corollary 5.12. *A division $\mathbb{Z}^n$-graded alternative algebra A over F is graded isomorphic to some $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ or to one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and $\mathbb{O}_4$ for some toral grading.*

If a division $\mathbb{Z}^n$-graded alternative algebra $A=\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n} A_{\boldsymbol{\alpha}}$ over F is a torus over F, then $A_{\mathbf{0}}=Z_{\mathbf{0}}=K=F$. So we have the following corollary which improves the result in [3], namely, the classification of alternative tori over any field.

Corollary 5.13. *Let A be an alternative torus over F. Then, A is graded isomorphic to either a quantum torus or the Cayley torus for some toral grading. If $\operatorname{ch} F=2$, then the Cayley torus does not exist; and so A is isomorphic to a quantum torus.*

Corollary 5.14. *Let A be an alternative torus over F and Z the centre. Assume that $A\neq Z$. Then A has degree 2 over Z iff $\operatorname{ch} F\neq 2$ and A is graded isomorphic to either the quaternion torus or the Cayley torus for some toral grading.*

Proof. It follows from Proposition 4.3 and Corollary 5.13. □

6. Division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebras

Let F in this section be a field of characteristic 0. Recall the definition of division $(\mathrm{A}_2,\mathbb{Z}^n)$-graded Lie algebras (see Definitions 2.5 and 2.7 in [2]): Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over F of type A_2, $\mathfrak{h}$ a split Cartan subalgebra of $\mathfrak{g}$, Δ its root system, and $G=(G,+,0)$ an abelian group. A Δ-graded Lie algebra $L=\bigoplus_{\mu\in\Delta\cup\{0\}} L_\mu$ over F with grading subalgebra $\mathfrak{g}=(\mathfrak{g},\mathfrak{h})$ is (A_2,G)-*graded* if $L=\bigoplus_{g\in G} L^g$ is a G-graded Lie algebra (assuming $\operatorname{supp} L$ generates G) such that $\mathfrak{g}\subset L^0$. Then L has the double grading, namely, $L=\bigoplus_{\mu\in\Delta\cup\{0\}}\bigoplus_{g\in G} L_\mu^g$, where $L_\mu^g=L_\mu\cap L^g$. Let $Z(L)$ be the centre of L and $\{h_\mu\in\mathfrak{h}\mid \mu\in\Delta\}$ the set of coroots. Then L is called a *division* (A_2,G)-*graded Lie algebra* if, for any $\mu\in\Delta$ and any $0\neq x\in L_\mu^g$, there exists $y\in L_{-\mu}^{-g}$ such that $[x,y]\equiv h_\mu$ modulo $Z(L)$.

Example 6.1. Let A be a unital alternative algebra over F. Let

$$psl_3(A) := \big(sl_3(F) \otimes_F A\big) \oplus D_{A,A},$$

where $D_{A,A}$ is the span of all inner derivations; i.e., $D_{A,A} = \operatorname{span}\{D_{a,b} \mid a, b \in A\}$, where $D_{a,b} = [L_a, L_b] + [R_a, R_b] + [L_a, R_b]$. ($L_a$ and R_a are the left and the right multiplication on A by a.) Then it is well known that $D_{A,A}$ is an ideal of the Lie algebra of derivations of A. Define a product $[\,,\,]$ on $psl_3(A)$ containing $D_{A,A}$ as the Lie subalgebra by

$$[x \otimes a, y \otimes b] = [x, y] \otimes \frac{ab + ba}{2} + \left(xy + yx - \frac{2}{3}\operatorname{tr}(xy)I\right) \otimes \frac{ab - ba}{2} + \frac{1}{3}\operatorname{tr}(xy)D_{a,b}$$

$$\text{and} \quad [D_{b,c}, x \otimes a] = x \otimes D_{b,c}(a) = -[x \otimes a, D_{b,c}]$$

for $x, y \in sl_3(F)$ and $a, b, c \in A$, where tr is the trace of a matrix and I is the identity 3×3 matrix. Then $psl_3(A)$ becomes a Lie algebra (see [13]). Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices in $sl_3(F)$, $\varepsilon_i : \mathfrak{h} \to F$ the projection onto the (i, i)-entry for $i = 1, 2, 3$, and $\Delta := \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, which is a root system of type A_2. Let $L_0 = (\mathfrak{h} \otimes_F A) \oplus D_{A,A}$ and $L_{\varepsilon_i - \varepsilon_j} = e_{ij}(F) \otimes_F A$, where $e_{ij}(F)$ is the set of matrix units with coefficients in F. Then $psl_3(A) = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha$ is a centreless A_2-graded Lie algebra with grading subalgebra $sl_3(F) = (sl_3(F), \mathfrak{h})$ (see [14] or [3]). Note that if A is associative, this Lie algebra is isomorphic to $psl_3(A)$ defined in [2]. Suppose that $A = \bigoplus_{g \in G} A_g$ is G-graded, then $psl_3(A) = \bigoplus_{g \in G} L^g$, where

$$L^g = (\mathfrak{h} \otimes_F A_g) \oplus \sum_{g = h + k} D_{A_h, A_k} \oplus \left(\bigoplus_{\varepsilon_i - \varepsilon_j \in \Delta} \big(e_{ij}(F) \otimes_F A_g\big)\right),$$

gives us an (A_2, G)-graded Lie algebra. We call this G-grading of $psl_3(A)$ the *natural compatible G-grading obtained from the G-grading of A*. Let $\{h_{ij} := e_{ij}(1) - e_{ji}(1) \mid \varepsilon_i - \varepsilon_j \in \Delta\}$ be the set of coroots. Then one can easily check that

$$\big[e_{ij}(1) \otimes a, e_{ji}(1) \otimes b\big] = h_{ij} \otimes 1 \quad \Longleftrightarrow \quad b = a^{-1}.$$

Thus, if A is division graded, then $psl_3(A)$ is a division (A_2, G)-graded Lie algebra.

The rest of arguments works similarly to the case $psl_3(A)$ for a unital associative algebra A. So we will omit some definitions and simply state results without proofs (see [2] for the detail).

Lemma 6.2. *Let A be a unital alternative algebra. Suppose that the A_2-graded Lie algebra $psl_3(A)$ described above is a division (A_2, G)-graded Lie algebra. Then A is a division G-graded algebra and the G-grading of $psl_3(A)$ is the natural compatible G-grading obtained from the G-grading of A.*

Proposition 6.3. *A division (A_2, G)-graded Lie algebra is an (A_2, G)-cover of $psl_3(A)$ admitting the natural compatible G-grading obtained from the G-grading of a division G-graded alternative algebra A.*

Thus, combining our main result Corollary 5.12, we obtain:

Theorem 6.4. *Any division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebra is an $(\mathrm{A}_2, \mathbb{Z}^n)$-cover of $psl_3(D_{\boldsymbol{\varphi},\boldsymbol{q}})$ for some $D_{\boldsymbol{\varphi},\boldsymbol{q}}$ or $psl_3(\mathbb{O}_i)$ for $i = 1, 2, 3, 4$. Conversely, any $psl_3(D_{\boldsymbol{\varphi},\boldsymbol{q}})$ or $psl_3(\mathbb{O}_i)$ is a division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebra.*

Acknowledgment

This is part of my Ph.D. thesis, written at the University of Ottawa. I would like to thank my supervisor, Professor Erhard Neher, for his encouragement and suggestions.

References

[1] K.A. Zhevlakov, A.M. Slinko, J.P. Shestakov, A.I. Shirshov, Rings That Are Nearly Associative, Academic Press, New York, 1982.
[2] Y. Yoshii, Root-graded Lie algebras with compatible grading, Comm. Algebra 29 (2001) 3365–3391.
[3] S. Berman, Y. Gao, Y. Krylyuk, E. Neher, The alternative tori and the structure of elliptic quasi-simple Lie algebras of type A_2, Trans. Amer. Math. Soc. 347 (1995) 4315–4363.
[4] B. Allison, Y. Gao, The root system and the core of an extended affine Lie algebra, Selecta Math. 7 (2001) 1–64.
[5] M. Boulagouaz, The graded and tame extensions, in: Commutative Ring Theory, in: Lecture Notes in Pure and Appl. Math., Vol. 153, Dekker, New York, 1994, pp. 27–40.
[6] D.S. Passman, Infinite Crossed Products, Academic Press, San Diego, 1989.
[7] J.C. McConnell, J.J. Pettit, Crossed products and multiplicative analogs of Weyl algebra, J. London Math. Soc. (2) 38 (1988) 47–55.
[8] S. Berman, Y. Gao, Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, J. Funct. Anal. 135 (1996) 339–389.
[9] K. McCrimmon, Nonassociative algebras with scalar involution, Pacific J. Math. 116 (1) (1985) 85–109.
[10] R.D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, New York, 1966.
[11] Y. Yoshii, Coordinate algebras of extended affine Lie algebras of type A_1, J. Algebra 234 (2000) 128–168.
[12] R.S. Pierce, Associative Algebras, in: Graduate Texts in Math., Vol. 88, Springer-Verlag, Berlin, 1982.
[13] G.B. Seligman, Rational Methods in Lie Algebras, in: Lecture Notes in Pure and Appl. Math., Vol. 17, Dekker, New York, 1976.
[14] S. Berman, R.V. Moody, Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy, Invent. Math. 108 (1992) 323–347.